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ON THE MULTIPLICITY OF NON-CONSTANT POSITIVE SOLUTIONS TO CERTAIN SEMI-LINEAR ELLIPTIC EQUATIONS

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Abstract. Many phenomena occurring in the natural environment have been modeled and studied using mathematical methods. In particular, investigating the existence and multiplicity of positive solutions, which represent the coexistence of equations, is always an intriguing research topic. To study the multiplicity of these positive solutions, it is necessary to analyze the behavior of positive solutions concerning a given parameter in the equation. In this research, we present a semi-linear partial differential equation to explain a series of natural phenomena through the study of positive solution behavior. We aim to investigate the existence and multiplicity of positive solutions that are not constant under homogeneous Neumann boundary conditions. Specifically, we apply the Mountain Pass theorem to demonstrate the existence of positive solutions for this equation, and further, we use the Leray-Schauder degree theory to explore sufficient conditions for the existence of two or more positive solutions.

1. Introduction

In this paper, we study the existence and multiplicity of non-constant positive solutions to the following semi-linear elliptic equation

$$\begin{cases} -\Delta u = (-D + \frac{\alpha u}{1+Mu})u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, and D, α, M are positive constants with the following hypothesis;

(H) $\alpha - DM > 0$.

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The research of the above elliptic problem plays an important role in the study of asymptotic behaviors to the prey-predator systems with Michaelis-Menten(or Holling-type II) functional response.

Observe that the equation $-D + \frac{\alpha u}{1+Mu} = 0$ has the unique positive root $u = \frac{D}{\alpha - DM}$ with respect to u since $\alpha - DM > 0$. For the sake of convenience, we denote this unique positive root by β, that is, $\beta = \frac{D}{\alpha - DM}$.

To state the main results of this paper, consider the following well-known eigenvalue problem

$$\begin{cases} -\Delta u = \mu u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Throughout this paper, we denote the eigenvalues of the above eigenvalue problem as μ_i and the respect multiplicity of μ_i as m_i for $i \geq 1$. It is well-known that $0 = \mu_1 < \mu_2 < \cdots$ and $\lim_{i\to\infty} = \infty$.

Now we state the main results of this paper.

THEOREM 1.1. *(i) If $m_2 < -D + \frac{\alpha}{M}$ and $\mu_2 < \frac{\alpha\beta}{(1+M\beta)^2}$, then (1.1) has at least one non-constant positive solution.*

(ii) If $\mu_2 < \alpha$ and $\frac{\alpha\beta}{(1+M\beta)^2} \in (\mu_{k_0}, \mu_{k_0+1})$ for some $k_0 \geq 2$, then (1.1) has at least two non-constant positive solutions.

2. Proofs of the main results

In this section, we prove the main results of this paper, Theorem 1.1.

By using the maximum principle, the hypothesis (H) gives that any solution u of the elliptic problem

$$\begin{cases} -\Delta u = (-D + \frac{\alpha u^+}{1+Mu^+})u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

is non-negative on $\overline{\Omega}$, where $u^+ = \max\{u, 0\}$. Thus it is easy to see that any solution of (2.1) is a non-negative solution of (1.1).

Define the function $J : H^1(\Omega) \to \mathbb{R}$ by

$$J(u) = \int_\Omega (\frac{1}{2}|\nabla u|^2 - G(u))dx,$$

where $G(u) = \int_0^u (-D + \frac{\alpha s^+}{1+Ms^+})sds$.

LEMMA 2.1. *J satisfies the Palais-Smale(PS) condition, that is, every sequence $\{u_n\} \subset H^1(\Omega)$ such that $J(u_n)$ is bounded for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} J'(u_n) = 0$, contains a convergent subsequence.*

Proof. Let $\{u_n\} \subset H^1(\Omega)$ be a sequence such that $|J(u_n)| \leq C$ and

$$\Big| \int_\Omega (\nabla u_n, \nabla\phi)dx - \int_\Omega \frac{\alpha u_n^+}{1+Mu_n^+} u_n \phi dx \Big| \leq \epsilon_n ||\phi||_{H^1} \tag{2.2}$$

for all $\phi \in H^1(\Omega)$, where C is a positive constant, $\lim_{n\to\infty} \epsilon_n = 0$ and $(\cdot,\cdot)$ is the $\mathbb{R}^n$-inner product. If u_n is bounded, then we see that the proof is completed. Contrariwise, suppose that $\lim_{n\to\infty} ||u_n||_\infty = \infty$(up to subsequnece). Let $z_n = \frac{u_n}{||u_n||_\infty}$. We may assume that, by taking a subsequence if necessary, $\lim_{n\to\infty} z_n = z_0$ strongly in $C(\Omega)$ and weakly in $H^1(\Omega)$ for some $z_0 \in H^1(\Omega)$ with $||z_0||_\infty = 1$.

Dividing the equation (2.2) with $\phi = z_0^- = \min\{z_0, 0\}$ by $||u_n||_\infty$, we have

(2.3)

$$\Big| \int_\Omega (\frac{\nabla u_n}{||u_n||_\infty}, \nabla z_0^-)dx - \int_\Omega (-D + \frac{\alpha u_n^+}{1+Mu_n^+}) \frac{u_n}{||u_n||_\infty} z_0^- dx \Big| \leq \epsilon_n \frac{||z_0^-||_{H^1}}{||u_n||_\infty}.$$

Since $\lim_{n\to\infty} z_n = z_0$ strongly in $C(\Omega)$, we know that

$$\lim_{n\to\infty} (\int_\Omega \nabla z_n \nabla z_0^- dx - \int_\Omega (-D + \frac{\alpha u^+}{1+Mu^+}) z_n z_0^- dx) = \int_\Omega |\nabla z_0^-|^2 dx + D \int_\Omega (z_0^-)^2 dx.$$

Since $\lim_{n\to\infty} \epsilon_n = 0$ in (2.3), it is concluded that

$$\int_\Omega |\nabla z_0^-|^2 dx + D \int_\Omega (z_0^-)^2 dx = 0,$$

and thus $z_0^- = 0$ which implies that $z_0 \geq 0$ on $\overline{\Omega}$. From the equation (2.2) and the given assumption, we have

$$\lim_{n\to\infty} \Big\{ \int_\Omega \nabla z_n \nabla\phi dx - \int_\Omega (-D + \frac{\alpha u^+}{1+Mu^+}) z_n \phi dx \Big\} = 0,$$

so that

$$\int_\Omega \nabla z_0 \nabla \phi dx - (-D + \frac{\alpha}{M}) \int_\Omega z_0 \phi dx = 0$$

for all $\phi \in H^1(\Omega)$, which gives that

$$\begin{cases} -\Delta z_0 = \alpha z_0 & \text{in } \Omega, \\ \frac{\partial z_0}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

By the maximum principle, we see that $z_0 > 0$ since $||z_0||_\infty = 1$ which derives a contradiction since $-D + \frac{\alpha}{M} > \mu_2$. Therefore we conclude that u_n is bounded, which completes the proof. □

LEMMA 2.2. *$J(0) = 0$ and there is a $\phi_0 \in H^1(\Omega) \backslash \overline{B}_\rho(0)$ with $J(\phi_0) \leq 0$, where $B_\rho(0)$ is an open ball with radius ρ centered at 0 in $H^1(\Omega)$.*

Proof. Note that $J(0) = 0$. It suffices to show that $\lim_{t\to\infty} J(t) = -\infty$.

For some large $t_0 > 0$, a small $\epsilon_0 > 0$ and some $\xi \in [0, t_0]$, we have

$$\begin{aligned}
\int_0^t (-D + \frac{\alpha s^+}{1+Ms^+})sds &= \int_0^t (-D + \frac{\alpha s}{1+Ms})sds \\
&= \int_0^t sf(s)ds \\
&= \int_0^{t_0} sf(s)ds + \int_{t_0}^t sf(s)ds \\
&= \int_0^{t_0} s^2\frac{f(s)-f(0)}{s}ds + \int_0^{t_0} sf(0)ds + \int_{t_0}^t sf(s)ds \\
&= \int_0^{t_0} s^2 f'(\xi)ds + f(0)\int_0^{t_0} sds + \int_{t_0}^t sf(s)ds \\
&\geq \{\min_{\xi\in[0,t_0]} f'(\xi)\}\int_0^{t_0} s^2ds - D\int_0^{t_0} sds + \int_{t_0}^t sf(s)ds \\
&\geq \frac{1}{3}\{\min_{\xi\in[0,t_0]} \frac{\alpha}{(1+M\xi)^2}\}{t_0}^3 - \frac{D}{2}{t_0}^2 + \int_{t_0}^t (-D + \frac{\alpha}{M} + \epsilon_0)sds \\
&\geq \frac{1}{3}\{\min_{\xi\in[0,t_0]} \frac{\alpha}{(1+M\xi)^2}\}{t_0}^3 - \frac{D}{2}{t_0}^2 + \frac{1}{2}(-D + \frac{\alpha}{M} + \epsilon_0)(t^2 - {t_0}^2),
\end{aligned}$$

where $f(s) = -D + \frac{\alpha s}{1+Ms}$. Therefore by taking the limit as $t \to \infty$, the assumptions gives

$$\begin{aligned}
\lim_{t\to\infty} J(t) &= -\lim_{t\to\infty}\int_\Omega (\int_0^t (-D + \frac{\alpha s^+}{1+Ms^+})sds)dx \\
&\leq -\lim_{t\to\infty}\int_\Omega \{\frac{1}{3}\{\min_{\xi\in[0,t_0]} \frac{\alpha}{(1+M\xi)^2}\}t_0^3 - \frac{D}{2}t_0^2 + \frac{1}{2}(-D + \frac{\alpha}{M} + \epsilon_0)(t^2 - t_0^2)\} \\
&= -\infty.
\end{aligned}$$

This completes the proof. □

LEMMA 2.3. *There exist positive constants ρ and r such that $J|_{\partial B_\rho}(0) \geq r$.*

Proof. For $\phi \geq 0$ and some $\xi \in [0, t_0]$, we have

$$\begin{aligned}
J(\phi) &= \int_\Omega \frac{1}{2}|\nabla\phi|^2 dx - \int_0^\phi (-D + \frac{\alpha s^+}{1+Ms^+}) s ds \\
&= \frac{1}{2}\int_\Omega |\nabla\phi|^2 dx - \int_\Omega \big(\int_0^\phi s^2 \frac{f(s)-f(0)}{s} ds + \int_0^\phi f(0) s ds\big) dx \\
&= \frac{1}{2}\int_\Omega |\nabla\phi|^2 dx - \int_\Omega \big(\int_0^\phi f'(\xi) s^2 ds\big) dx - f(0)\int_\Omega \big(\int_0^\phi s ds\big) dx \\
&\geq -f(0)\int_\Omega \frac{1}{2}\phi^2 dx - \frac{1}{3}\{\max_{\xi\in[0,\phi]} f'(\xi)\}\int_\Omega \phi^3 dx \\
&\geq D\int_\Omega \frac{1}{2}\phi^2 dx - \frac{1}{3}\{\max_{\xi\in[0,\phi]} \frac{\alpha}{(1+M\xi)^2}\}\int_\Omega \phi^3 dx \\
&\geq \frac{\min\{1, D\}}{2}||\phi||^2_{H^1} - \frac{1}{3}\{\max_{\xi\in[0,\phi]} \frac{\alpha}{(1+M\xi)^2}\}||\phi||^3_{L^3},
\end{aligned}$$

where $f(s) = -D + \frac{\alpha s}{1+Ms}$.

On the other hand, it follows from Lemma 7.2 in [5] and Lemma 5.4 in [1] that $||\phi||_{L^3} \leq K_1 \phi^{\frac{1}{2}} (\frac{5}{2})^{\frac{5}{6}} ||\phi||_{H^1}$, and thus we have

$$J(\phi) \geq ||\phi||^2_{H^1}(C_1 - C_2||\phi||_{H^1})$$

for some positive constants C_1 and C_2. Thus for sufficiently small ϕ, we conclude that $C_1 - C_2||\phi||_{H^1} > 0$, which completes the proof. □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. (i) By Mountain pass theorem in [3], J has a (mountain pass type; mp-type) critical point $u_* \geq r > 0$. Furthermore, we may assume that

$$C_q(J, u_*) \cong \delta_{q,1} G, \tag{2.4}$$

where $C_q(J, u_*)$ is the qth-critical group with the abelian coefficient group G of J at u_*, $q = 0, 1, 2, \cdots$ and $\delta_{q,1}$ is the krouecker delta.(For more details, one can refer to [4].)

Obviously, 0 and β are critical points of J. To show $u_* \not\equiv 0$ and $u_* \not\equiv \beta$, we claim that the critical groups of $u_*, 0, \beta$ are all different. Note that for all $u, \phi, \psi \in H^1(\Omega)$,

$$< J'(u), \phi > = \int_\Omega (\nabla u \nabla\phi - (-D + \frac{\alpha u}{1+Mu}) u\phi) dx,$$

$$< J''(u)\phi, \psi > = \int_\Omega (\nabla\phi\nabla\psi - (-D + \frac{\alpha u}{1+Mu} + \frac{\alpha u}{(1+Mu)^2})\phi\psi) dx,$$

where $< \cdot, \cdot >$ denotes the $H^1(\Omega)$-inner product. We see that

$$< J''(u)\phi, \psi >= \int_\Omega (\nabla\phi\nabla\psi + D\phi\psi)dx,$$

and thus

$$< J''(u)\phi, \phi >= \int_\Omega (|\nabla\phi|^2 + D\phi^2)dx > 0,$$

for all $\phi \in H^1(\Omega)$ with $\phi \neq 0$. This implies that 0 is a non-degenerate critical point of J, moreover the Morse index of 0, which will be denoted by $M - ind(J, 0)$, is zero; and $C_q(J, 0) \cong \delta_{q,0}G$ by Theorem 4.1 in [4, Chapter 1]. Therefore $C_1(J, 0) \cong 0$. On the other hand, since $< J''(\beta)\phi, \psi >= \int_\Omega (\nabla\phi\nabla\psi - \frac{\alpha\beta}{(1+M\beta)^2})\phi\psi)dx$, it follows from the given assumption $\mu_2 < \frac{\alpha\beta}{(1+M\beta)^2}$ that

$$< J''(\beta)\phi, \phi >= \int_\Omega (|\nabla\phi|^2 - \frac{\alpha\beta}{(1+M\beta)^2})\phi^2)dx \leq (\mu_2 - \frac{\alpha\beta}{(1+M\beta)^2}) \int_\Omega \phi^2 dx < 0$$

for $\phi \neq 0$ in the subspace of $H^1(\Omega)$ spanned by the eigenfunctions corresponding to eigenvalues $\mu_k (k \leq 2)$. Thus the Morse index of β is at least 2, that is,

$$J^* = M - ind(J, \beta) \geq 2.$$

By the shifting theorem(Theorem 5.4) in [4, Chapter 1],

$$C_q(J, \beta) \cong C_q - J^*(\tilde{J}, \beta) \tag{2.5}$$

for all $q = 0, 1, 2, \cdots$, where $\tilde{J}$ is the restriction of J to a certain manifold of $H^1(\Omega)$ whose dimension is the nullity of the Hessian $J''(\beta)$. Therefore if $q = 1$ in (2.5), then $1 - J^* \notin (0, dim(ker J''(\beta)))$ which gives $C_{1-J^*}(\tilde{J}, \beta) \cong 0$, and thus

$$C_1(J, \beta) \cong 0.$$

However, it follows from (2.4) that

$$C_1(J, u^*) \ncong 0.$$

Hence we see that $u_* \not\equiv 0$ and $u_* \not\equiv \beta$, which implies that u_* is a nonconstant positive solution of (1.1). This completes the proof of (i).

(ii) Define the operator

$$T(u) = (-\Delta + P)^{-1}((-D + \frac{\alpha u}{1 + Mu}) + Pu),$$

where P is a positive constant such that $(-D + \frac{\alpha u}{1+Mu}) + Pu$ is increasing in $u \in [0, \infty]$. It follows from the standard regularity theorem and Sobolev embedding theorem that T is a compact mapping from $C(\overline{\Omega})$

into itself. Now suppose that $0, \beta, u_*$ are the only critical points of J. Recall that u_* is an isolated critical point of mo-type. Applying Theorem 2 in [6], we have

$$\deg_W(I - T, B_r(u_*) \cap W, 0) = -1,$$

where W is the natural positive cone in $C(\overline{\Omega})$ and $B_r(u_*)$ is the ball of radius r centered at u_* in $C(\overline{\Omega})$ for sufficiently small $r > 0$. Moreover, by Lemma 13.1 and Lemma 13.4 in [2], it follows from the hypotheses (H) that

$$\deg_W(I - T, B_r(0) \cap W, 0) = 1,$$
$$\deg_W(I - T, B_R(0) \cap W, 0) = 0$$

for sufficiently small $r > 0$ and sufficiently large $R > 0$. Using the Leray-Schauder formula, we have

$$\deg_W(I - T, B_r(\beta) \cap W, 0) = (-1)^{\sum_{k=2}^{k_0} m_k}$$

since $\frac{\alpha\beta}{(1+M\beta)^2} \in (\mu_{K_0}, \mu_{K_0+1})$ for some $k_0 \geq 2$. By the additivity property of the degree, we have

$$\begin{aligned} 0 &= \deg_W(I - T, B_R(0) \cap W, 0) \\ &= \deg_W(I - T, B_r(0) \cap W, 0) + \deg_W(I - T, B_r(u_*) \cap W, 0) \\ &\quad + \deg_W(I - T, B_r(\beta) \cap W, 0) \\ &= 1 + (-1) + (-1)^{\sum_{k=2}^{k_0} m_k} \\ &= (-1)^{\sum_{k=2}^{k_0} m_k}, \end{aligned}$$

which derives a contradiction. This completes the proof. □

References

[1] A. Adams, *Soblev spaces*, Academic Press, New York, 1975.
[2] H. Amann, *Fixed point equations and nonlinear eigenvalue problems in ordered banach spaces*, SIAM Rev. 18 (1976), **4**, 620-709.
[3] A. Ambrosetti and P. H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal., **14** (1973), 349-381.
[4] K. C. Chang, *Infinite dimensional Morse theory and multiple solutions problems*, Birkhäuser, Boston, 1993.
[5] D. Gilbarg, N. S. Trudinger, *Elliptic partial differential equations of second order*, Springer-Verlag, New York, 1983.
[6] H. Hofer, *A note on the Topological degree at a critical point of mountain pass type*, Proc. Amer. Soc., **90** (1984), 309-315.

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